Author Meets Critics: Responses

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January 8, 2020

Thanks

Scott Weinstein

Hilbert: reliability

Scott writes:

(John) seeks clarification and understanding of mathematical topics locally through the medium of formalization of specific mathematical theories,
whereas Hilbert sought verification and justification of mathematics globally through through the formalization of an overarching theory into which all of mathematics could be specified.

I respond:

1. Hilbert sought clarification and understanding of mathematical topics locally (as well) through the medium of formalization of specific mathematical theories: witness geometry, arithmetic
2. Hilbert had formally retired before Gödel showed his second aim was impossible (if one holds that the justification must be finitistic).

Hilbert: autonomy

Scott writes:

(John) may not identify with . . . ‘Hilbert’s embrace of the autonomy of mathematical knowledge

I respond:

1. I do embrace the autonomy of mathematical knowledge. My faith is the same as Hilbert’s.
   However, in the wake of Gödel, I think one should adopt a more open system and apply formalization to particular topics.
2. A strange (to me) feature of my work is the observation (not original) that Euclidean geometry, real and complex algebraic geometry, are all finitistically consistent.

Syntactic Character I

Scott quotes me:

Without pettifoggery on the exact meaning on the exact meaning of syntactic, . . . conditions on cardinality of Stone spaces and topology of Stone Spaces play crucial roles . . .

Theorem (Vaught)

The following conditions are equivalent:

1. *Each $S_n(T)$ is countable;*
2. *T has a countable, saturated model;*

Syntactic Character II

One is tempted to say, by analogy with the discussion in the last paragraph of 3 [where he asserts that the condition: finitely many n-types for each n is syntactic], that condition (numbered 1.1 here) is purely syntactical. Indeed, in 1.1, no reference to any semantical concept, such as "model", is made. However, a little thought convinces one that a notion of "purely syntactical condition" wide enough to include Theorem 1 would be so broad as to be pointless. (Vaught, 1959)

I respond:

Syntactic is a strong word. But the basic notions of model theory are grounded at a very basic level; well within second order arithmetic. So Vaught seemed too conservative and I wanted to clarify his instinct that a wider notion is needed.

Properties of classes of theories

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

Stability theory: Local – Global

Shelah's key distinction

1. Local properties concern a single formula: ϕ is NOP (stable), NIP, SOP_n etc. But we may require the each formula to have the property
   Generally speaking these are π^0_2-properties.
   Stable theories have a good notion of dimension locally.
2. Global properties involve the interaction of many formulas: ω-stable, superstable, NOTOP, NDOP.
   Generally speaking this are Π^1_1 properties.
   And the result is the ability to assign (countable trees of invariants) that determine models.

Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

Formalization: not stability theory

Formalization is the focal point of modern logic.

Arguments for the effect of formal logic on other areas of mathematics can (and should) be made for set theory, computability theory, and proof theory.

But I don't have the expertise.

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1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
   General epistemological goals: accessibility, clarity, flexibility
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Naturality/Complexity in Mathematics

Both are term often employed by mathematicians. But there is no standard meaning.

Naturality

Binary operations that are not associative are often seen to be unnatural or at best ‘combinatorial’.

Eudoxus versus Dedekind
Maybe ‘computability’ can substitute for ‘naturality’.

Complexity

1. computability hierarchies
2. decomposability
3. classification

Relations among Model Theory, Computability Theory, and Proof Theory

My general impression had been

There aren't any

Example: There are continuum many 'trivial' strongly minimal theories. Model theoretically, they all behave the same. But only countably many are recursively axiomatized.

Weinstein's program

In contrast, Scott's approach to calibrate proof theoretic, and thus certain type of combinatorial, strength of the assertion:

'T is at point x in the stability hierarchy'

is very intriguing.

A striking interaction of model theory and computability theory

Theorem and Corollary

Trivial strongly minimal theories are model complete after naming constants.
All countable models of a trivial, strongly minimal theory with at least one computable model are $0''$-decidable, and that the spectrum of computable models of any trivial, strongly minimal theory is Σ^0_5.
(Goncharov, Harizanov, Laskowski, Lempp, and McCoy)

Tim Bays

Structuralism???

Structuralism, Putman, Benaceraff

There is no mention of the philosophical study of structuralism in my book.
That is because I see the notion of structure as fully and completely explicated in set theory. Note that the same structure can be treated in any logic.

Thus, I am pleased to see Tim's:
'The puzzle is: given that we already knew that mathematicians only cared about structures up to isomorphism, why did anyone worry about the kinds of indeterminacy of reference issue issue raised by the model-theoretic argument. The moral, of course is that we shouldn't worry. Good mathematics doesn't care which object counts as the number 2; good philosophy shouldn't care either.'

Structure versus logic

Definition

1. A *vocabulary* τ is a list of function, constant and relation symbols.
2. A τ*-structure* $\mathcal{A} = \langle A, R_1, \ldots R_n \rangle$ is a set A (the domain of $\mathcal{A}$) with an interpretation of each symbol in τ. That is, for each n-ary relation symbol R in τ, R^A is a collection of n-tuples from A.
3. A structure is *many-sorted* if there are a family of unary predicates T_i and each the variable, function, and relation symbols assign sorts to their arguments and, in the case of functions, values.

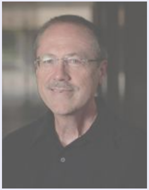
Whether we quantify over elements or subsets of various orders depends on our choice of logic.

The (many-sorted) structure is the same. It is defined in the most basic set theory.

Tim asked: What is the target?

No target: Question A

Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?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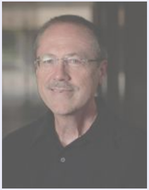

Detlefsen

Detlefsen:

Of course as a model theorists, I thought
Categoricity means uncountable categoricity of a first order theory

Detlefsen also asked

Question B

Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how do we justify the claim that categoricity is or would be a virtue just in just former?

What is virtue?

What is virtue?

I take ' a virtuous property' to be one which has significant mathematical consequences for a theory or its models. Thus, a better property of theories has more mathematical consequences for the theory.

Is categoricity virtuous?

I argued categoricity of a second order theory does not, by itself, shed any mathematical light on the categorical structure.

But that particular axiomatizations (Peano, Dedekind) were virtuous.

However, categoricity in power for first order and infinitary logic yields significant structural information about models of theory.

This kind of structural analysis leads to a fruitful classification theory for complete first order theories.
Indeed, fewer models usually indicates a better structure theory for models of the theory.

Clarifying: What is virtue?

Paul McEldowney argues that such a preemptory proclamation of the nature of virtue should be replaced by more systematic analysis.

1. **Purpose** What is the purpose of the theory? E.G. rigor, description, organization, or
   Instrumentalism: Formal theories serve to extend our knowledge regarding a class $\mathcal{W}$ of τ-structures of which we have some prior scientific interest.
2. **Virtue** Given purpose $\mathcal{P}$, a property Φ of a theory T is virtuous insofar as it promotes T's ability to achieve $\mathcal{P}$. E.G.
   Instrumental virtue: Assuming instrumentalism, a property Φ of T is virtuous insofar as it promotes T's function as a knowledge-extending instrument regarding a class $\mathcal{W}$ of τ-structures.

I think my notion provides a specific kind of instrumental virtue. This analysis gives a framework to study other virtues.

Dividing lines

A property is a dividing line if both it and its negation are virtuous.
Plato: Cut through the Middle! (The Statesman)

But, further, there is a positive and a negative side to dividing line.
E.g. The stability is hierarchy is *internally successful* because the positive answers lead to a useful structure theory (dimension, decomposition of models, applications)

Morley's conjecture

$I(\kappa, T)$ is nondecreasing except possibly from $\aleph_0$ to $\aleph_1$.

Shelah's strategy

Find a series of dividing lines such that at each stage P_i:

1. If P_i holds, T has the maximal number of models in every $\kappa > \aleph_0$.
2. If P_i fails we have a better idea of the structure of the models.

After 4 steps if $\neg P_4(T)$ we can determine each model of T by a tree of invariants of countable models.

The resultant 13 possible spectrum functions are all non-decreasing.

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The remarkable point is that these dividing lines have ramifications in number theory, combinatorics, group theory, differential equations, etc.

Methodological questions

1. How does this analysis inform our understanding of the interaction of concepts in various areas of mathematics?
2. How general is the dividing line strategy; Shelah has applied it several times with different guiding questions and resulting different dividing lines?
3. Can one characterize what sorts of problems are appropriate for a dividing line stategy? If your answer is 'classification', what sorts of classification can be expected to have such fruitful applications across mathematics?
4. Does McEldowney's analysis provide a framework for examining such strategies?

Colin McLarty

Rigor

Mclarty objects to viewing model theorists as more rigorous than mathematicians at large.

1. I did not mean to assert that; in my view model theorists are mathematicians. As group they have no particular claim to greater rigor or reliability.
2. What I did mean to assert to *philosophers* is that passing by inductive definition from
    1. $\mathbb{N} = \{0, 1, 2, 3, 4 \ldots\}$ to
    2. $(\mathbb{N}, S)$ to
    3. $(\mathbb{N}, S, +)$ to
    4. $(\mathbb{N}, S, +, \times)$ to
    5. $(\mathbb{N}, S, +, \times, 2^x)$

    is passing from kindergarden to rocket science.
    Mathematicians know these are different objects.

Isomorphism

In the book: Fix a vocabulary τ: An isomorphism is bijection between two τ structures that preserves the relations and functions of τ.

Mclarty

Mclarty asserts: In mainstream mathematics today there is one notion: Namely, an isomorphism is a morphism with an inverse.

Response

I have to disagree with this. It assumes that mainstream mathemathics revolves around a category theory notion of morphism. This is true of a large very influential area of mathematics; but far from the entire mainstream.

I should have said: There are many other notions of isomorphism in mathematics.

From notion to definition

the other notion

An (iso) morphism preserves structure

Gowers: Princeton Companion to Math

'to give a definition of isomorphism of Riemann surfaces':
'two Riemann surfaces X and Y are (conformally or holomorphically) equivalent if there is a topological equivalence between them which preserves the geometry (i.e. a homeomorphism that preserves the angles between curves, or takes holomorphic functions to holomorphic functions or takes rational functions to rational functions. (These conditions are all equivalent.)'

In order to define the notion of isomorphism; one must fix the concepts that are being studied –as in making a formalization.

Algebraic geometry

McLarty: Are fields a natural focus for model theory.
Response: Algebraic geometry is because it is the study of solutions of (positive Boolean combinations) of equations.

Hrushovski-Van den Dries-Weil theorem

A. Weil showed that a birational group law that is only partially defined can be extended to an algebraic group.

Van den Dries gives a model theoretic proof that extends to 'differentially algebraic' group chunks.

Hrushovski gives abstract conditions (4th generation of Hilbert's defining of the field in Euclidean geometry) for the existence of a group or field.

In particular: Any definable group in an algebraically closed field is an *algebraic group*.

Ties across mathematics

What property do the following objects share?

1. A pure set
2. natural numbers with successor
3. a divisible Abelian group
4. an algebraically closed field
5. the solution set in a differentially closed filed of the order three algebraic differential equation over $\mathbb{Q}$ satisfied by the analytic Weirsrass j-function

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Answer

They are strongly minimal sets.
The first and last are $\aleph_0$ categorical with trivial geometry.

To complete the geometrical classification, Baldwin-Paolini: Strongly minimal Steiner systems.

Formalism freeness

> *That mathematics is practiced in what one might call a formalism-free manner has always been the case – and remains the case. Of course, no one would have thought to put it this way prior to the emergence of the foundational formal systems in the late nineteenth and early twentieth centuries; . . . Kennedy - On Formalism Freeness*

This book principally aims to show how formalizing (**syntax and semantics** particular areas of mathematics can illuminate both mathematics and its philosophy.
Both Juliette and Colin emphasize that mathematics is not done that way. Colin argues for a Euclid- Hilbert I -Bourbaki style axiomatization of mathematics through category theory.

Recent work of Lieberman, Rosický, and Vasey carries out model theory in that spirit (but on ZFC foundations).

May many flowers bloom!